CHAPTER 9
FUNDAMENTALS OF ALGEBRA

The numbers and operating rules of arithmetic form a part of a very important branch of mathematics called ALGEBRA.

Algebra extends the concepts of arithmetic so that it is possible to generalize the rules for operating with numbers and use these rules in manipulating symbols other than numbers. It does not involve an abrupt change into a distinctly new field, but rather provides a smooth transition into many branches of mathematics with a continuation of knowledge already gained in basic arithmetic.

The idea of expressing quantities in a general way, rather than in the specific terms of arithmetic, is fairly common. A typical example is the formula for the perimeter of a rectangle, $P = 2L + 2W$, in which the letter P represents perimeter, L represents length, and W represents width. It should be understood that $2L = 2(L)$ and $2W = 2(W)$. If the L and the W were numbers, parentheses or some other multiplication sign would be necessary, but the meaning of a term such as 2L is clear without additional signs or symbols.

All formulas are algebraic expressions, although they are not always identified as such. The letters used in algebraic expressions are often referred to as LITERAL NUMBERS (literal implies "letteral").

Another typical use of literal numbers is in the statement of mathematical laws of operation. For example, the commutative, associative, and distributive laws, introduced in chapter 3 with respect to arithmetic, may be restated in general terms by the use of algebraic symbols.

COMMUTATIVE LAWS

The word "commutative" is defined in chapter 3. Remember that the commutative laws refer to those situations in which the factors and terms of an expression are rearranged in a different order.

ADDITION

The algebraic form of the commutative law for addition is as follows:

$$a + b = b + a$$

From this law, it follows that

$$a + (b + c) = a + (c + b) = (c + b) + a$$

In words, this law states that the sum of two or more addends is the same regardless of the order in which the addends are arranged.

The arithmetic example in chapter 3 shows only one specific numerical combination in which the law holds true. In the algebraic example, a, b, and c represent any numbers we choose, thus giving a broad inclusive example of the rule. (Note that once a value is selected for a literal number, that value remains the same wherever the letter appears in that particular example or problem. Thus, if we give a the value of 12, in the example just given, a's value is 12 wherever it appears.)

MULTIPLICATION

The algebraic form of the commutative law for multiplication is as follows:

$$ab = ba$$

In words, this law states that the product of two or more factors is the same regardless of the order in which the factors are arranged.

ASSOCIATIVE LAWS

The associative laws of addition and multiplication refer to the grouping (association) of terms and factors in a mathematical expression.

ADDITION

The algebraic form of the associative law for addition is as follows:

$$a + b + c = (a + b) + c = a + (b + c)$$

In words, this law states that the sum of three or more addends is the same regardless of the manner in which the addends are grouped.

MULTIPLICATION

The algebraic form of the associative law for multiplication is as follows:

$$a \cdot b \cdot c = (a \cdot b) \cdot c = a \cdot (b \cdot c)$$

In words, this law states that the product of three or more factors is the same regardless of the manner in which the factors are grouped.

DISTRIBUTIVE LAW

The distributive law refers to the distribution of factors among the terms of an additive expression. The algebraic form of this law is as follows:

$$a(b + c) = ab + ac$$

From this law, it follows that: If the sum of two or more quantities is multiplied by a third quantity, the product is found by applying the multiplier to each of the original quantities separately and summing the resulting expressions.

ALGEBRAIC SUMS

The word "sum" has been used several times in this discussion, and it is important to realize the full implication where algebra is concerned. Since a literal number may represent either a positive or a negative quantity, a sum of several literal numbers is always understood to be an ALGEBRAIC SUM. That is, it is the sum that results when the algebraic signs of all the addends are taken into consideration.

The following problems illustrate the procedure for finding an algebraic sum:

Let $a = 3$, $b = -2$, and $c = 4$.

$$\text{Then } a + b + c = (3) + (-2) + (4)$$
$$= 5$$

$$\text{Also, } a - b - c = a + (-b) + (-c)$$
$$= 3 + (+2) + (-4)$$
$$= 1$$

The second problem shows that every expression containing two or more terms to be combined by addition and subtraction may be rewritten as an algebraic sum, all negative signs being considered as belonging to specific terms and all operational signs being positive.

It should be noted, in relation to this subject, that the laws of signs for algebra are the same as those for arithmetic.

ALGEBRAIC EXPRESSIONS

An algebraic expression is made up of the signs and symbols of algebra. These symbols include the Arabic numerals, literal numbers, the signs of operation, and so forth. Such an expression represents one number or one quantity. Thus, just as the sum of 4 and 2 is one quantity, that is, 6, the sum of c and d is one quantity, that is, $c + d$. Likewise $\frac{a}{b}$, $\sqrt{b}$, ab, $a - b$, and so forth, are algebraic expressions each of which represents one quantity or number.

Longer expressions may be formed by combinations of the various signs of operation and the other algebraic symbols, but no matter how complex such expressions are they still represent one number. Thus the algebraic expression $\frac{-a + \sqrt{2a + b}}{6} - c$ is one number

The arithmetic value of any algebraic expression depends on the values assigned to the literal numbers. For example, in the expression $2x^2 - 3ay$, if $x = -3$, $a = 5$, and $y = 1$, then we have the following:

$$2x^2 - 3ay = 2(-3)^2 - 3(5)(1)$$
$$= 2(9) - 15 = 18 - 15 = 3$$

Notice that the exponent is an expression such as $2x^2$ applies only to the x. If it is desired to indicate the square of 2x, rather than 2 times the square of x, then parentheses are used and the expression becomes $(2x)^2$.

Practice problems. Evaluate the following algebraic expressions when $a = 4$, $b = 2$, $c = 3$, $x = 7$, and $y = 5$. Remember, the order of operation is multiplication, division, addition, and subtraction.

1. $3x + 7y - c$
2. $xy - 4a^2$
3. $\frac{ax}{b} + y$
4. $c + \frac{ay^2}{b}$

Answers:

1. 53
2. -29
3. 19
4. 53

TERMS AND COEFFICIENTS

The terms of an algebraic expression are the parts of the expression that are connected by plus and minus signs. In the expression $3abx + cy - k$, for example, $3abx$, cy, and k are the terms of the expression.

An expression containing only one term, such as 3ab, is called a monomial (mono means one). A binomial contains two terms; for example, $2r + by$. A trinomial consists of three terms. Any expression containing two or more terms may also be called by the general name, polynomial (poly means many). Usually special names are not given to polynomials of more than three times. The expression $x^3 - 3x^2 + 7x + 1$ is a polynomial of four terms. The trinomial $x^2 + 2x + 1$ is an example of a polynomial which has a special name.

Practice problems. Identify each of the following expressions as a monomial, binomial, trinomial, or polynomial. (Some expressions may have two names.)

1. x	3. abx	5. $3y^2 + 4$
2. $3y + a + b$	4. $4 + 2b + y + z$	6. $\frac{2y}{6} + 1$

Answers:

1. Monomial	2. Trinomial (also polynomial)
3. Monomial	4. Polynomial
5. Binomial (also polynomial)	6. Binomial (also polynomial)

In general, a COEFFICIENT of a term is any factor or group of factors of a term by which the remainder of the term is to be multiplied. Thus in the term 2axy, 2ax is the coefficient of y, 2a is the coefficient of xy, and 2 is the coefficient of axy. The word "coefficient" is usually used in reference to that factor which is expressed in Arabic numerals. This factor is sometimes called the NUMERICAL COEFFICIENT. The numerical coefficient is customarily written as the first factor of the term. In 4x, 4 is the numerical coefficient, or simply the coefficient, of x. Likewise, in $24xy^2$, 24 is the coefficient of xy^2 and in $16(a + b)$, 16 is the coefficient of $(a + b)$. When no numerical coefficient is written it is understood to be 1. Thus in the term xy, the coefficient is 1.

COMBINING TERMS

When arithmetic numbers are connected by plus and minus signs, they can always be combined into one number. Thus,

$$5 - 7\frac{1}{2} + 8 = 5\frac{1}{2}$$

Here three numbers are added algebraically (with due regard for sign) to give one number. The terms have been combined into one term.

Terms containing literal numbers can be combined only if their literal parts are the same. Terms containing literal factors in which the same letters are raised to the same power are called like terms. For example, 3y and 2y are like terms since the literal parts are the same. Like terms are added by adding the coefficients of the like parts. Thus, $3y + 2y = 5y$ just as 3 bolts + 2 bolts = 5 bolts. Also $3a^2b$ and a^2b are like; $3a^2b + a^2b = 4a^2b$ and $3a^2b - a^2b = 2a^2b$. The numbers ay and by are like terms with respect to y. Their sum could be indicated in two ways: $ay + by$ or $(a + b)y$. The latter may be explained by comparing the terms to denominate numbers. For instance, a bolts + b bolts = (a + b) bolts.

Like terms are added or subtracted by adding or subtracting the numerical coefficients and placing the result in front of the literal factor, as in the following examples:

$$7x^2 - 5x^2 = (7 - 5)x^2 = 2x^2$$

$$5b^2x - 3ay^2 - 8b^2x + 10ay^2 = -3b^2x + 7ay^2$$

Dissimilar or unlike terms in an algebraic expression cannot be combined when numerical values have not been assigned to the literal factors. For example, $-5x^2 + 3xy - 8y^2$ contains three dissimilar terms. This expression cannot be further simplified by combining terms through addition or subtraction. The expression may be rearranged as $x(3y - 5x) - 8y^2$ or $y(3x - 8y) - 5x^2$, but such a rearrangement is not actually a simplification.

Practice problems. Combine like terms in the following expression:

1. $2a + 4a$	4. $2ay^2 - ay^2$
2. $y + y^2 + 2y$	5. $bx^2 + 2bx^2$
3. $4\frac{ay}{c} - \frac{ay}{c}$	6. $2y + y^2$

Answers:

1. $6a$	4. ay^2
2. $y^2 + 3y$	5. $3bx^2$
3. $3\frac{ay}{c}$	6. $2y + y^2$

SYMBOLS OF GROUPING

Often it is desired to group two or more terms to indicate that they are to be considered and treated as though they were one term even though there may be plus and minus signs between them. The symbols of grouping are parentheses ( ) (which we have already used), brackets [ ], braces { }, and the vinculum ___. The vinculum is sometimes called the "overscore." The fact that -7 + 2 - 5 is to be subtracted from 15, for example, could be indicated in any one of the following ways:

$$15 - (-7 + 2 - 5)$$
$$15 - [-7 + 2 - 5]$$
$$15 - \{-7 + 2 - 5\}$$
$$15 - \overline{-7 + 2 - 5}$$

Actually the vinculum is seldom used except in connection with a radical sign, such as in $\sqrt{a + b}$, or in a Boolean algebra expression. Boolean algebra is a specialized kind of symbolic notation which is discussed in Mathematics, Volume 3, NavPers 10073.

Parentheses are the most frequently used symbols of grouping. When several symbols are needed to avoid confusion in grouping, parentheses usually comprise the innermost symbols, followed by brackets, and then by braces as the outermost symbols. This arrangement of grouping symbols is illustrated as follows:

$$2x - \{3y + [- 8 - 5y - (x - 4)]\}$$

REMOVING AND INSERTING GROUPING SYMBOLS

Discussed in the following paragraphs are various rules governing the removal and insertion of parentheses, brackets, braces, and the vinculum. Since the rules are the same for all grouping symbols, the discussion in terms of parentheses will serve as a basis for all.

Removing Parentheses

If parentheses are preceded by a minus sign, the entire quantity enclosed must be regarded as a subtrahend. This means that each term of the quantity in parentheses is subtracted from the expression preceding the minus sign. Accordingly, parentheses preceded by a minus sign can be removed, if the signs of all terms within the parentheses are changed.

This may be explained with an arithmetic example. We recall that to subtract one number from another, we change the sign of the subtrahend and proceed as in addition. To subtract -7 from 16, we change the sign of -7 and proceed as in addition, as follows:

$$\begin{aligned} 16 - (-7) &= 16 + 7 \\ &= 23 \end{aligned}$$

It is sometimes easier to see the result of changing signs in the subtrahend if the minus sign preceding the parentheses is regarded as a multiplier. Thus, the thought process in removing parentheses from an expression such as - (4 - 3 + 2) would be as follows: Minus times plus is minus, so the first term of the expression with parentheses removed is - 4. (Remember that the 4 in the original expression is understood to be a +4, since it has no sign showing.) Minus times minus is plus, so the second term is +3. Minus times plus is minus, so the third term is -2. The result is - 4 + 3 - 2, which reduces to -3.

This same result can be reached just as easily, in an arithmetic expression, by combining the numbers within the parentheses before applying the negative sign which precedes the parentheses. However, in an algebraic expression with no like terms such combination is not possible. The following example shows how the rule for removal of parentheses is applied to algebraic expressions:

$$2a - (-4x + 3by) = 2a + 4x - 3by$$

Parentheses preceded by a plus sign can be removed without any other changes, as the following example shows:

$$2b + (a - b) = 2b + a - b = a + b$$

Many expressions contain more than one set of parentheses, brackets, and other symbols of grouping. In removing symbols of grouping, it is possible to proceed from the outside inward or from the inside outward. For the beginner, it is simpler to start on the inside and work toward the outside, collecting terms and simplifying as one proceeds. In the following example the inner grouping symbols are removed first:

$$2a - [x + (x - 3a) - (9a - 5x)]$$
$$= 2a - [x + x - 3a - 9a + 5x]$$
$$= 2a - [7x - 12a]$$
$$= 2a - 7x + 12a$$
$$= 14a - 7x$$

Enclosing Terms in Parentheses

When it is desired to enclose a group of terms in parentheses, the group of terms remains unchanged if the sign preceding the parentheses is positive. This is illustrated as follows:

$$3x - 2y + 7x - y = (3x - 2y) + (7x - y)$$

Note that this agrees with the rule for removing parentheses preceded by a plus sign.

If terms are enclosed within parentheses preceded by a minus sign, the signs of all the terms enclosed must be changed as in the following example:

$$3x - 2y + 7x - y = 3x - (2y - 7x + y)$$

Practice problems. In problems 1 through 4, remove the symbols of grouping and combine like terms. In problems 5 through 8, enclose the first two terms in parentheses preceded by a plus sign (understood) and the last two in parentheses preceded by a minus sign.

1. $6a - (4a - 3)$
2. $3x + [2x - 4y(6 - 4x)] + 2y - (3 - x + 3y)$
3. $-a + [-a - (2a + 3)] + 3$
4. $(7x - 3ay) - (4a - b) + 16$
5. $4a - 3b - 2c + 4d$
6. $-2 -3x +4y - z$
7. $x + 4y + 3z + 7$
8. $-4 + 2a - 6c + 3d$

Answers:

1. $2a + 3$
2. $6x + 16xy - 25y - 3$
3. $-4a$
4. $7x - 3ay - 4a + b + 16$
5. $(4a - 3b) - (2c - 4d)$
6. $(-2 -3x) - (-4y + z)$
7. $(x + 4y) - (-3z - 7)$
8. $(-4 + 2a) - (6c - 3d)$

EXPONENTS AND RADICALS

Exponents and radicals have the same meaning in algebra as they do in arithmetic. Thus, if n represents any number then $n^2 = n \cdot n$, $n^3 = n \cdot n \cdot n$, etc. By the same reasoning, n^m means that n is to be taken as a factor m times. That is, n^m is equal to $n \cdot n \cdot n \cdot \ldots$, with n appearing m times. The series of dots, called ellipsis (not to be confused with the geometric figure having a similar name, ellipse), represents continuation of the same pattern or the same symbol.

The rules of operation with exponents are also the same in algebra as in arithmetic. For example, $n^2 \cdot n^3 = n^{2+3} = n^5$. Some care is necessary to avoid confusion over an expression such as $3^2 \cdot 3^3$. In this example, n = 3 and the product desired is 3^5, not 9^5. In general, $3^a \cdot 3^b = 3^{a+b}$, and a similar result is reached whether the factor which acts as a base for the exponents is a number or a letter. Thus the general form can be expressed as follows:

$$n^a \cdot n^b = n^{a+b}$$

In words, the general rule for multiplication involving exponents is as follows: When multiplying terms whose literal factors are like, the exponents are added. This rule may be applied to problems involving division, if all expressions containing exponents in denominators are rewritten as expressions with negative exponents. For example, the fraction $\frac{x^2y}{xy^2}$ can be rewritten as $(x^2y)(x^{-1}y^{-2})$, which is equal to $(x^{2-1})(y^{1-2})$. This reduces to xy^{-1}, or $\frac{x}{y}$. Notice that the result is the same as it would have been if we had simply subtracted the exponents of literal factors in the denominator from the exponents of the same literal factors in the numerator.

The algebraic rules for radicals also remain the same as those of arithmetic. In arithmetic, $\sqrt{4} = 4^{1/2} = 2$. Likewise, in algebra $\sqrt{a} = a^{1/2}$ and $\sqrt[n]{a} = a^{1/n}$.

MULTIPLYING MONOMIALS

If a monomial such as 3abc is to be multiplied by a numerical multiplier, for example 5, the coefficient alone is multiplied, as in the following example:

$$5 \times 3abc = 15abc$$

When the numerical factor is not the initial factor of the expression, as in x(2a), the result of the multiplication is not written as x2a. Instead, the numerical factor is interchanged with literal factors by use of the commutative law of multiplication. The literal factors are usually interchanged to place them in alphabetical order, and the final result is as follows:

$$x(2a) = 2ax$$

The rule for multiplication of monomials may be stated as follows: Multiply the numerical coefficients to form the coefficient of the product. Multiply the literal factors, combining exponents in like factors, to form the literal part of the product. The complete process is illustrated in the following example:

$$\begin{aligned}(2ab)(3a^2)(2b^3) &= 12a^{1+2}b^{1+3}\\ &= 12a^3b^4\end{aligned}$$

Practice problems. Perform the indicated operations:

1. $(2x^2)(5x^5)$
2. $(-5ab^2)(2a^2b)$
3. $(-4x^4y)(-3xy^4)$
4. $(2^a)(2^b)$
5. $(-4a^3)^2$
6. $(3a^2b)^2$

Answers:

1. $10x^7$
2. $-10a^3b^3$
3. $12x^5y^5$
4. 2^{a+b}
5. $16a^6$
6. $9a^4b^2$

DIVIDING MONOMIALS

As may be expected, the process of dividing is the inverse of multiplying. Because $3 \times 2a = 6a$, $6a \div 3 = 2a$, or $6a \div 2 = 3a$. Thus, when the divisor is numerical, divide the coefficient of the dividend by the divisor.

When the divisor contains literal parts that are also in the dividend, cancellation may be performed as in arithmetic. For example, $6ab \div 3a$ may be written as follows:

$$\frac{(2)(3a)(b)}{3a}$$

Cancellation of the common literal factor, 3a, from the numerator and denominator leaves 2b as the answer for this division problem.

When the same literal factors appear in both the divisor and the dividend, but with different exponents, cancellation may still be used, as follows:

$$\begin{aligned}\frac{14a^3b^3x}{-21a^2b^5x} &= \frac{(7)(2)a^2ab^3x}{(7)(-3)a^2b^3b^2x}\\ &= \frac{2a}{-3b^2} = -\frac{2a}{3b^2}\end{aligned}$$

This same problem may be solved without thinking in terms of cancellation, by rewriting with negative exponents as follows:

$$\begin{aligned}\frac{14a^3b^3x}{-21a^2b^5x} &= \frac{2a^{3-2}b^{3-5}x^{1-1}}{-3}\\ &= \frac{2ab^{-2}}{-3} = \frac{2a}{-3b^2}\\ &= -\frac{2a}{3b^2}\end{aligned}$$

Practice problems. Perform the indicated operations:

1. $\frac{x^5}{x^6}$
2. $\frac{a^9b^4}{a^6b^3}$
3. $\frac{a^2bc^2}{abc}$
4. $\frac{a^2b}{ab^2}$
5. $\sqrt{16x^4y^6}$
6. $\sqrt{x^{4a}y^{2a}}$
7. $\frac{5a^4b}{10a^2b^3}$
8. $\frac{10x^2y^3z^4}{-5xy^2z^3}$
9. $\sqrt{100a^8b^4}$
10. $\sqrt{a^{6}b^{6n}}$

Answers:

1. x^{-1}
2. a^3b
3. ac
4. $\frac{a}{b}$
5. $\pm 4x^2y^3$
6. $\pm x^{2a}y^a$
7. $\frac{a^2}{2b^2}$
8. $-2xyz$
9. $\pm 10a^4b^2$
10. $\pm a^3b^{3n}$

OPERATIONS WITH POLYNOMIALS

Adding and subtracting polynomials is simply the adding and subtracting of their like terms. There is a great similarity between the operations with polynomials and denominate numbers. Compare the following examples:

1. Add 5 qt and 1 pt to 3 qt and 2 pt.

$$\begin{array}{r} 3 \text{ qt} + 2 \text{ pt} \\ \underline{5 \text{ qt} + 1 \text{ pt}} \\ 8 \text{ qt} + 3 \text{ pt} \end{array}$$

2. Add $5x + y$ to $3x + 2y$.

$$\begin{array}{r} 3x + 2y \\ \underline{5x + y} \\ 8x + 3y \end{array}$$

One method of adding polynomials (shown in the above examples) is to place like terms in columns and to find the algebraic sum of the like terms. For example, to add $3a + b - 3c$, $3b + c - d$, and $2a + 4d$, we would arrange the polynomials as follows:

$$\begin{array}{rrrr} 3a & + \; b & - \; 3c & \\ & 3b & + \; c & - \; d \\ \underline{2a} & & & \underline{+ \; 4d} \\ 5a & + \; 4b & - \; 2c & + \; 3d \end{array}$$

Subtraction may be performed by using the same arrangement—that is, by placing terms of the subtrahend under the like terms of the minuend and carrying out the subtraction with due regard for sign. Remember, in subtraction the signs of all the terms of the subtrahend must first be mentally changed and then the process completed as in addition. For example, subtract $10a + b$ from $8a - 2b$, as follows:

$$\begin{array}{r} 8a - 2b \\ \underline{10a + b} \\ -2a - 3b \end{array}$$

Again, note the similarity between this type of subtraction and the subtraction of denominate numbers.

Addition and subtraction of polynomials also can be indicated with the aid of symbols of grouping. The rule regarding changes of sign when removing parentheses preceded by a minus sign automatically takes care of subtraction. For example, to subtract $10a + b$ from $8a - 2b$, we can use the following arrangement:

$$\begin{aligned} (8a - 2b) - (10a + b) &= 8a - 2b - 10a - b \\ &= -2a - 3b \end{aligned}$$

Similarly, to add $-3x + 2y$ to $-4x - 5y$, we can write

$$\begin{aligned} (-3x + 2y) + (-4x - 5y) &= -3x + 2y - 4x - 5y \\ &= -7x - 3y \end{aligned}$$

Practice problems. Add as indicated, in each of the following problems:

1. $\begin{array}{r} 3a + b \\ \underline{2a + 5b} \end{array}$
2. $(6s^3t + 3s^2t + st + 5) + (s^3t - 5)$
3. $4a + b + c$, $a + c - d$, and $3a + 2b + 2c$
4. $\begin{array}{l} 4x + 2y \\ 3x - y + z \\ \underline{x - z} \end{array}$

In problems 5 through 8, perform the indicated operations and combine like terms.

5. $(2a + b) - (3a + 5b)$
6. $(5x^3y + 3x^2y) - (x^3y)$
7. $(x + 6) + (3x + 7)$
8. $(4a^2 - b) - (2a^2 + b)$

Answers:

1. $5a + 6b$	5. $-(a + 4b)$
2. $7s^3t + 3s^2t + st$	6. $4x^3y + 3x^2y$
3. $8a + 3b + 4c - d$	7. $4x + 13$
4. $8x + y$	8. $2(a^2 - b)$

MULTIPLICATION OF A POLYNOMIAL BY A MONOMIAL

We can explain the multiplication of a polynomial by a monomial by using an arithmetic example. Let it be required to multiply the binomial expression, $7 - 2$, by 4. We may write this $4 \times (7 - 2)$ or simply $4(7 - 2)$. Now $7 - 2 = 5$. Therefore, $4(7 - 2) = 4(5) = 20$. Now, let us solve the problem a different way. Instead of subtracting first and then multiplying, let us multiply each term of the expression by 4 and then subtract. Thus, $4(7 - 2) = (4 \times 7) - (4 \times 2) = 20$. Both methods give the same result. The second method makes use of the distributive law of multiplication.

When there are literal parts in the expression to be multiplied, the first method cannot be used and the distributive method must be employed. This is illustrated in the following examples:

$$4(5 + a) = 20 + 4a$$
$$3(a + b) = 3a + 3b$$
$$ab(x + y - z) = abx + aby - abz$$

Thus, to multiply a polynomial by a monomial, multiply each term of the polynomial by the monomial.

Practice problems. Multiply as indicated:

1. $2a(a - b)$
2. $4a^2(a^2 + 5a + 2)$
3. $-4x(-y - 3z)$
4. $2a^3(a^2 - ab)$

Answers:

1. $2a^2 - 2ab$
2. $4a^4 + 20a^3 + 8a^2$
3. $4xy + 12xz$
4. $2a^5 - 2a^4b$

MULTIPLICATION OF A POLYNOMIAL BY A POLYNOMIAL

As with the monomial multiplier, we explain the multiplication of a polynomial by a polynomial by use of an arithmetic example. To multiply $(3 + 2)(6 - 4)$, we could do the operation within the parentheses first and then multiply, as follows:

$$(3 + 2)(6 - 4) = (5)(2) = 10$$

However, thinking of the quantity $(3 + 2)$ as one term, we can use the method described for a monomial multiplier. That is, we can multiply each term of the multiplicand by the multiplier, $(3 + 2)$, with the following result:

$$(3 + 2)(6 - 4) = [(3 + 2) \times 6 - (3 + 2) \times 4]$$

Now considering each of the two resulting products separately, we note that each is a binomial multiplied by a monomial.

The first is

$$(3 + 2)6 = (3 \times 6) + (2 \times 6)$$

and the second is

$$\begin{aligned} -(3 + 2)4 &= -[(3 \times 4) + (2 \times 4)] \\ &= -(3 \times 4) - (2 \times 4) \end{aligned}$$

Thus we have the following result:

$$\begin{aligned} (3 + 2)(6 - 4) &= (3 \times 6) + (2 \times 6) \\ &\quad - (3 \times 4) - (2 \times 4) \\ &= 18 + 12 - 12 - 8 \\ &= 10 \end{aligned}$$

The complete product is formed by multiplying each term of the multiplicand separately by each term of the multiplier and combining the results with due regard to signs.

Now let us apply this method in two examples involving literal numbers.

1. $(a + b)(m + n) = am + an + bm + bn$
2. $(2b + c)(r + s + 3t - u) = 2br + 2bs + 6bt - 2bu + cr + cs + 3ct - cu$

The rule governing these examples is stated as follows: The product of any two polynomials is found by multiplying each term of one by each term of the other and adding the results algebraically.

It is often convenient, especially when either of the expressions contains more than two terms, to place the polynomial with the fewer terms beneath the other polynomial and multiply term by term beginning at the left. Like terms of the partial products are placed one beneath the other to facilitate addition.

Suppose we wish to find the product of $3x^2 - 7x - 9$ and $2x - 3$. The procedure is

$$\begin{array}{rrrr} 3x^2 & - 7x & - 9 & \\ 2x & - 3 & & \\ \hline 6x^3 & - 14x^2 & - 18x & \\ & - 9x^2 & + 21x & + 27 \\ \hline 6x^3 & - 23x^2 & + 3x & + 27 \end{array}$$

Practice problems. In the following problems, multiply and combine like terms:

1. $(2a - 3)(a + 2)$
2. $(ax + b)(ax - b)$
3. $\begin{array}{l} x^3 + 5x^2 - x + 2 \\ \underline{2x + 3} \end{array}$
4. $\begin{array}{l} 2a^2 + 5ab - b^2 \\ \underline{a + b} \end{array}$

Answers:

1. $2a^2 + a - 6$
2. $a^2x^2 - b^2$
3. $2x^4 + 13x^3 + 13x^2 + x + 6$
4. $2a^3 + 7a^2b + 4ab^2 - b^3$

SPECIAL PRODUCTS

The products of certain binomials occur frequently. It is convenient to remember the form of these products so that they can be written immediately without performing the complete multiplication process. We present four such special products as follows, and then show how each is derived:

1. Product of the sum and difference of two numbers.

EXAMPLE: $(x - y)(x + y) = x^2 - y^2$

2. Square the sum of two numbers.

EXAMPLE: $(x + y)^2 = x^2 + 2xy + y^2$

3. Square of the difference of two numbers.

EXAMPLE: $(x - y)^2 = x^2 - 2xy + y^2$

4. Product of two binomials having a common term.

EXAMPLE: $(x + a)(x + b) = x^2 + (a + b)x + ab$

Product of Sum and Difference

The product of the sum and difference of two numbers is equal to the square of the first number minus the square of the second number. If, for example, x - y is multiplied by x + y, the middle terms cancel one another. The result is the square of x minus the square of y, as shown in the following illustration:

$$\begin{array}{llll} x & - y & & \\ \underline{x} & \underline{+ y} & & \\ x^2 & - xy & & \\ \underline{} & \underline{+ xy} & \underline{- y^2} & \\ x^2 & & - y^2 & \end{array}$$

By keeping this rule in mind, the product of the sum and difference of two numbers can be written down immediately by writing the difference of the squares of the numbers. For example, consider the following three problems:

$$(x + 3)(x - 3) = x^2 - 3^2 = x^2 - 9$$

$$(5a + 2b)(5a - 2b) = (5a)^2 - (2b)^2 = 25a^2 - 4b^2$$

$$(7x + 4y)(7x - 4y) = 49x^2 - 16y^2$$

RATIONALIZING DENOMINATORS.— The product of the sum and difference of two numbers is useful in rationalizing a denominator that is a binomial. For example, in a fraction such as

$$\frac{2}{\sqrt{2} - 6}$$

the denominator can be altered so that no radical terms appear in it. (This process is called rationalizing.) The denominator must be multiplied by $\sqrt{2} + 6$, which is called the conjugate of $\sqrt{2} - 6$. Since the value of the original fraction would be changed if we multiplied only the denominator, our multiplier must be applied to both the numerator and the denominator. Multiplying the original fraction by

$$\frac{\sqrt{2} + 6}{\sqrt{2} + 6}$$

is, in effect, the same as multiplying it by 1.

The result of rationalizing the denominator of this fraction is as follows:

$$\begin{aligned} \frac{2}{\sqrt{2} - 6} \cdot \frac{\sqrt{2} + 6}{\sqrt{2} + 6} &= \frac{2(\sqrt{2} + 6)}{(\sqrt{2})^2 - 6^2} \\ &= \frac{2(\sqrt{2} + 6)}{2 - 36} \\ &= \frac{2(\sqrt{2} + 6)}{2(1 - 18)} \\ &= \frac{2(\sqrt{2} + 6)}{2(-17)} \\ &= \frac{\sqrt{2} + 6}{-17} \end{aligned}$$

MENTAL MULTIPLICATION.—The product of the sum and difference can be utilized to mentally multiply two numbers that differ from a multiple of 10 by the same amount, one greater and the other less. For example, 67 is 3 less than 70 while 73 is 3 more than 70. The product of 67 and 73 is then found as follows:

$$\begin{aligned} 67(73) &= (70 - 3)(70 + 3) \\ &= 70^2 - 3^2 = 4{,}900 - 9 = 4{,}891 \end{aligned}$$

Square of Sum or Difference

The square of the SUM of two numbers is equal to the square of the first number plus twice the product of the numbers plus the square of the second number. The square of the DIFFERENCE of the same two numbers has the same form, except that the sign of the middle term is negative.

These results are evident from multiplication. When x and y represent the two numbers, we obtain

$$\begin{array}{lll} x + y & \qquad & x - y \\ \underline{x + y} & & \underline{x - y} \\ x^2 + xy & & x^2 - xy \\ \underline{\quad + xy + y^2} & & \underline{\quad - xy + y^2} \\ x^2 + 2xy + y^2 & & x^2 - 2xy + y^2 \end{array}$$

Applying this rule to the squares of the binomials 3a + 2b and 3a - 2b, we have the following two cases:

1. $(3a + 2b)^2 = (3a)^2 + 2(3a)(2b) + (2b)^2$
   $= 9a^2 + 12ab + 4b^2$

2. $(3a - 2b)^2 = 9a^2 - 12ab + 4b^2$

The square of the sum or difference of two numbers is applicable to squaring a binomial that contains one or two irrational terms, as in the following examples:

1. $(\sqrt{3} + 8)^2 = (\sqrt{3})^2 + 2(8)(\sqrt{3}) + 64$
   $= 3 + 16\sqrt{3} + 64 = 67 + 16\sqrt{3}$

2. $(\sqrt{3} - 8)^2 = (\sqrt{3})^2 - 2(8)(\sqrt{3}) + 64$
   $= 3 - 16\sqrt{3} + 64 = 67 - 16\sqrt{3}$

3. $(\sqrt{5} + \sqrt{7})^2 = (\sqrt{5})^2 + 2\sqrt{5}\sqrt{7} + (\sqrt{7})^2$
   $= 5 + 2\sqrt{35} + 7 = 12 + 2\sqrt{35}$

4. $(\sqrt{5} - \sqrt{7})^2 = 12 - 2\sqrt{35}$

The square of the sum or difference of two numbers can be applied to the process of mentally squaring certain numbers. For example, 82^2 can be expressed as $(80 + 2)^2$ while 67^2 can be expressed as $(70 - 3)^2$. We find that

$$\begin{aligned} (80 + 2)^2 &= 80^2 + 2(80)(2) + 2^2 \\ &= 6{,}400 + 320 + 4 = 6{,}724 \end{aligned}$$

$$\begin{aligned} (70 - 3)^2 &= 70^2 - 2(70)(3) + 3^2 \\ &= 4{,}900 - 420 + 9 = 4{,}489 \end{aligned}$$

Binomials Having a Common Term

The binomials x + 2 and x - 3 have a common term, x. They have two unlike terms, +2 and -3. The product of these binomials is

$$\begin{array}{l} x + 2 \\ \underline{x - 3} \\ x^2 + 2x \\ \underline{\quad - 3x - 6} \\ x^2 - x - 6 \end{array}$$

Inspection of this product shows that it is obtained by squaring the common term, adding the sum of the unlike terms multiplied by the common term, and finally adding the product of the unlike terms.

Apply this rule to the product of 3y - 5 and 3y + 4. The common term is 3y; its square is $9y^2$. The sum of the unlike terms is -5 + 4 = -1; the sum of the unlike terms multiplied by the common term is -3y; and the product of the unlike terms is -5(4) = -20. The product of the two binomials is

$$(3y - 5)(3y + 4) = 9y^2 - 3y - 20$$

The product of two binomials having a common term is applicable to the multiplication of numbers like $\sqrt{3} + 7$ and $\sqrt{3} - 2$ which contain irrational terms. For example,

$$\begin{aligned} (\sqrt{3} + 7)(\sqrt{3} - 2) &= (\sqrt{3})^2 + 5\sqrt{3} - 14 \\ &= 3 + 5\sqrt{3} - 14 \\ &= -11 + 5\sqrt{3} \end{aligned}$$

Practice problems. In problems 1 through 4, multiply and combine terms. In 5 through 8, simplify by using special products.

1. $(x + 4)(x + 2)$
2. $(\sqrt{a} - b)^2$
3. $(7a + 4b)(7a - 4b)$
4. $(ax + y)^2$
5. $\dfrac{2}{\sqrt{2} - 2}$
6. $48(52)$
7. $(\sqrt{3} + 7)^2$
8. $(73)^2$

Answers:

1. $x^2 + 6x + 8$
2. $a - 2b\sqrt{a} + b^2$
3. $49a^2 - 16b^2$
4. $a^2x^2 + 2axy + y^2$
5. $-(\sqrt{2} + 2)$
6. $(50 - 2)(50 + 2) = 2496$
7. $52 + 14\sqrt{3}$
8. $(70 + 3)(70 + 3) = 5329$

DIVISION OF A POLYNOMIAL BY A MONOMIAL

Division, like multiplication, may be distributive. Consider, for example, the problem $(4 + 6 - 2) \div 2$, which may be solved by adding the numbers within the parentheses and then dividing the total by 2. Thus,

$$\frac{4 + 6 - 2}{2} = \frac{8}{2} = 4$$

Now notice that the problem may also be solved distributively.

$$\begin{aligned}\frac{4 + 6 - 2}{2} &= \frac{4}{2} + \frac{6}{2} - \frac{2}{2}\\ &= 2 + 3 - 1\\ &= 4\end{aligned}$$

CAUTION: Do not confuse problems of the type just described with another type which is similar in appearance but not in final result. For example, in a problem such as $2 \div (4 + 6 - 2)$ the beginner is tempted to divide 2 successively by 4, then 6, and then -2, as follows:

$$\frac{2}{4 + 6 - 2} \neq \frac{2}{4} + \frac{2}{6} - \frac{2}{2}$$

Notice that we have canceled the "equals" sign, because $2 \div 8$ is obviously not equal to $1/2 + 2/6 - 1$. The distributive method applies only in those cases in which several different numerators are to be used with the same denominator

When literal numbers are present in an expression, the distributive method must be used, as in the following two problems:

1. $$\begin{aligned}\frac{2ax + aby + a}{a} &= \frac{2ax}{a} + \frac{aby}{a} + \frac{a}{a}\\ &= 2x + by + 1\end{aligned}$$

2. $$\begin{aligned}\frac{18ab^2 - 12bc}{6b} &= \frac{18ab^2}{6b} - \frac{12bc}{6b}\\ &= 3ab - 2c\end{aligned}$$

Quite often this division may be done mentally, and the intermediate steps need not be written out.

DIVISION OF A POLYNOMIAL BY A POLYNOMIAL

Division of one polynomial by another proceeds as follows:

1. Arrange both the dividend and the divisor in either descending or ascending powers of the same letter.
2. Divide the first term of the dividend by the first term of the divisor and write the result as the first term of the quotient.
3. Multiply the complete divisor by the quotient just obtained, write the terms of the product under the like terms of the dividend, and subtract this expression from the dividend.
4. Consider the remainder as a new dividend and repeat steps 1, 2, and 3.

EXAMPLE:

$$(10x^3 - 7x^2y - 16xy^2 + 12y^3) \div (5x - 6y)$$

SOLUTION:

$$\begin{array}{r|l} & 2x^2 + xy - 2y^2 \\ \hline 5x - 6y & 10x^3 - 7x^2y - 16xy^2 + 12y^3 \\ & \underline{10x^3 - 12x^2y} \\ & \qquad 5x^2y - 16xy^2 \\ & \qquad \underline{5x^2y - 6xy^2} \\ & \qquad\qquad -10xy^2 + 12y^3 \\ & \qquad\qquad \underline{-10xy^2 + 12y^3} \end{array}$$

In the example just shown, we began by dividing the first term, $10x^3$, of the dividend by the first term, $5x$, of the divisor. The result is $2x^2$. This is the first term of the quotient.

Next, we multiply the divisor by $2x^2$ and subtract this product from the dividend. Use the remainder as a new dividend. Get the second term, xy, in the quotient by dividing the first term, $5x^2y$, of the new dividend by the first term, 5x, of the divisor. Multiply the divisor by xy and again subtract from the dividend.

Continue the process until the remainder is zero or is of a degree lower than the divisor. In the example being considered, the remainder is zero (indicated by the double line at the bottom). The quotient is $2x^2 + xy - 2y^2$.

The following long division problem is an example in which a remainder is produced:

$$
\begin{array}{r|l}
 & x^2 - x + 3 \\ \hline
x + 3 & x^3 + 2x^2 \qquad + 5 \\
 & \underline{x^3 + 3x^2} \\
 & \quad - x^2 \\
 & \quad \underline{- x^2 - 3x} \\
 & \qquad\qquad 3x + 5 \\
 & \qquad\qquad \underline{3x + 9} \\
 & \qquad\qquad\quad - 4
\end{array}
$$

The remainder is -4.

Notice that the term -3x in the second step of this problem is subtracted from zero, since there is no term containing x in the dividend. When writing down a dividend for long division, leave spaces for missing terms which may enter during the long division process.

In arithmetic, division problems are often arranged as follows, in order to emphasize the relationship between the remainder and the divisor:

$$\frac{5}{2} = 2 + \frac{1}{2}$$

This same type of arrangement is used in algebra. For example, in the problem just shown, the results could be written as follows:

$$\frac{x^3 + 2x^2 + 5}{x + 3} = x^2 - x + 3 - \frac{4}{x + 3}$$

Remember, before dividing polynomials arrange the terms in the dividend and divisor according to either descending or ascending powers of one of the literal numbers. When only one literal number occurs, the terms are usually arranged in order of descending powers.

For example, in the polynomial $2x^2 + 4x^3 + 5 - 7x$ the highest power among the literal terms is x^3. If the terms are arranged according to descending powers of x, the term in x^3 should appear first. The x^3 term should be followed by the x^2 term, the x term, and finally the constant term. The polynomial arranged according to descending powers of x is $4x^3 + 2x^2 - 7x + 5$.

Suppose that $4ab + b^2 + 15a^2$ is to be divided by $3a + 2b$. Since 3a can be divided evenly into $15a^2$, arrange the terms according to descending powers of a. The dividend takes the form

$$15a^2 + 4ab + b^2$$

Synthetic Division

Synthetic division is a shorthand method of dividing a polynomial by a binomial of the form x - a. For example, if $3x^4 + 2x^3 + 2x^2 - x - 6$ is to be divided by x - 1, the long form would be as follows:

$$
\begin{array}{r|l}
 & \qquad 3x^3 + 5x^2 + 7x + 6 \\ \hline
x - 1 & 3x^4 + 2x^3 + 2x^2 - x - 6 \\
 & \underline{3x^4 - 3x^3} \\
 & \qquad + 5x^3 + 2x^2 \\
 & \qquad \underline{+ 5x^3 - 5x^2} \\
 & \qquad\qquad + 7x^2 - x \\
 & \qquad\qquad \underline{+ 7x^2 - 7x} \\
 & \qquad\qquad\qquad + 6x - 6 \\
 & \qquad\qquad\qquad \underline{\underline{+ 6x - 6}}
\end{array}
$$

Notice that every alternate line of work in this example contains a term which duplicates the one above it. Furthermore, when the subtraction is completed in each step, these duplicated terms cancel each other and thus have no effect on the final result. Another unnecessary duplication results when terms from the dividend are brought down and rewritten prior to subtraction. By omitting these duplications, the work may be condensed as follows:

$$
\begin{array}{r|rrrrr}
 & & 3x^3 & +5x^2 & +7x & +6 \\ \hline
x - 1 & 3x^4 & +2x^3 & +2x^2 & -x & -6 \\
 & & -3x^3 & -5x^2 & -7x & -6 \\ \cline{2-6}
 & & +5x^3 & +7x^2 & +6x & 0
\end{array}
$$

The coefficients of the dividend and the constant term of the divisor determine the results of each successive step of multiplication and subtraction. Therefore, we may condense still further by writing only the nonliteral factors, as follows:

$$\begin{array}{r|rrrrr}
 & 3 & +5 & +7 & +6 & \\ \hline
-1 & 3 & +2 & +2 & -1 & -6 \\
 & & -3 & -5 & -7 & -6 \\ \hline
 & 3 & +5 & +7 & +6 & 0
\end{array}$$

Notice that if the coefficient of the first term in the dividend is brought down to the last line, then the numbers in the last line are the same as the coefficients of the terms in the quotient. Thus we do not really need to write a separate line of coefficients to represent the quotient. Instead, we bring down the first coefficient of the dividend and make the subtraction "subtotals" serve as coefficients for the rest of the quotient, as follows:

$$\begin{array}{r|rrrrr}
x - 1 & 3 & 2 & 2 & -1 & -6 \\
 & & -3 & -5 & -7 & -6 \\ \hline
 & 3 & 5 & 7 & 6 & 0
\end{array}$$

The unnecessary writing of plus signs is also eliminated here.

The use of synthetic division is limited to divisors of the form $x - a$, in which the degree of x is 1. Thus the degree of each term in the quotient is 1 less than the degree of the corresponding term in the dividend. The quotient in this example is as follows:

$$3x^3 + 5x^2 + 7x + 6$$

The sequence of operations in synthetic division may be summarized as follows, using as an example the division of $3x - 4x^2 + x^4 - 3$ by $x + 2$:

$$\begin{array}{r|rrrrr}
2 & 1 & 0 & -4 & 3 & -3 \\
 & & 2 & -4 & 0 & 6 \\ \hline
 & 1 & -2 & 0 & 3 & -9
\end{array}$$

First, rearrange the terms of the dividend in descending powers of x. The dividend then becomes $x^4 - 4x^2 + 3x - 3$, with 1 understood as the coefficient of the first term. No x^3 term appears in the polynomial, but we supply a zero as a place holder for the x^3 position.

Second, bring down the 1 and multiply it by the +2 of the divisor. Place the result under the zero, and subtract. Multiply the result (-2) by the +2 of the divisor, place the product under the -4 of the dividend, and subtract. Continue this process, finally obtaining $x^3 - 2x^2 + 3$ as the quotient. The remainder is -9.

Practice problems. In the following problems, perform the indicated operations. In 4, 5, and 6, first use synthetic division and then check your work by long division:

1. $(a^3 - 3a^2 + a) \div a$
2. $\dfrac{x^6 - 7x^5 + 4x^4}{x^2}$
3. $(10x^3 - 7x^2y - 16xy^2 + 12y^3) \div (2x^2 + xy - 2y^2)$
4. $(x^2 + 11x + 30) \div (x + 6)$
5. $(12 + x^2 - 7x) \div (x - 3)$
6. $(a^2 - 11a + 30) \div (a - 5)$

Answers:

1. $a^2 - 3a + 1$
2. $x^4 - 7x^3 + 4x^2$
3. $5x - 6y$
4. $x + 5$
5. $x - 4$
6. $a - 6$